A Generalization of the Binet-Minc Formula for the Evaluation of Permanents

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ABSTRACT

A formula for the sum of the coefficients of monomials of the form $x_{l_1}^{j_1} \cdots x_{l_p}^{j_p}$ is given, where $j_1, \dots, j_p$ are given positive integers, in the polynomial

$$\prod_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} x_j \right).$$

When $p = n$ and $j_1 = \cdots j_n = 1$, this formula coincides with the Binet-Minc formula for the evaluation of the permanent of the matrix (a_{ij}).

1. INTRODUCTION

Let $A = (a_{ij})$ be an $n \times m$ matrix. We consider the polynomial

$$\prod_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} x_j \right). \tag{1}$$

The coefficients of such polynomials were investigated, more than a century ago, by Hammond [5] and Muir [8] (cf. the bibliography of Minc [6]). The sum of the coefficients of monomials of the form $x_{l_1} \cdots x_{l_n}$, where $(l_1 \dots l_n)$ extends over all n-permutations of the set $M = \{1, \dots, m\}$, has been called

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the permanent of the matrix A. Recently Bebiano [3] treated the sum of the coefficients of the form $x_1^{j_1} \cdots x_m^{j_m}$, where j_i are nonnegative integers whose sum is n, and obtained an identity expressing (1) as a polynomial with coefficients involving permanents induced by A. She used the identity to deduce in a simple way the Binet-Minc [7] and the Ryser formulas for the evaluation of permanents.

In this paper we consider the sum of the coefficients of monomials of the form $x_{l_1}^{j_1} \cdots x_{l_p}^{j_p}$, where $j_1, \dots, j_p$ are positive integers whose sum is n and $(l_1 \dots l_p)$ extends over all p-permutations of M, and give a Binet-Minc type formula for the sum of the coefficients. Our results are given in Theorems 2 and 3. The proof of Theorem 2 uses Theorem 1, which states the cardinality of the set of partitions of the set $N = \{1, \dots, n\}$ corresponding to a partition of an integer n. As an application of these theorems, two expressions for the Stirling numbers are given in Section 4.

2. NOTATION AND TERMINOLOGY

We use some notation and terminology following Aigner [1] and Andrews [2]. Let M and N be the following sets of integers: $M = \{1, \dots, m\}$ and $N = \{1, \dots, n\}$. Let π denote a partition of the set N into p blocks, i.e. p disjoint nonempty subsets, $N_1, \dots, N_p$. The lattice L_N is the totality of partitions of N with natural semiorder, which is denoted by "$\leqslant$" or "$\geqslant$". The set N itself is the unique maximal, I, of L_N, and the unique minimal, 0, is the partition into n distinct one point sets (cf. [1, p. 13]).

Corresponding to π, the cardinalities of the blocks $N_1, \dots, N_p$ arranged in nonincreasing order, say, $j_1 \geqslant \cdots \geqslant j_p$, form a partition of the integer n into p parts, which is denoted by $j(p) = (j_1 \cdots j_p)$. The set of partitions of the integer n into p parts is denoted by $G(p)$, and the union of $G(1), \dots, G(n)$ is denoted by G_n. On G_n there is a natural semiorder relation corresponding to L_N. For $p = q + 1$, $j(p) = (j_1 \cdots j_p) \in G(p)$ is said to be *covered* by $k(q) = (k_1 \dots k_q) \in G(q)$ when one element in $k(q)$ is a sum of two in $j(p)$, and the remaining $q - 1$ elements in $j(p)$ and $k(q)$ both form the same set of integers. The semiorder in G_n is called the dominance order in G_n (cf. [1, p. 18]). The semiorder in G_n is denoted by the same notation "$\leqslant$" or "$\geqslant$" as the one in L_N. When $n \geqslant 5$, G_n cannot be a lattice, since $(n-2\,1\,1)$ and $(n-3\,2\,1)$ are covered by $(n-1\,1)$ and $(n-2\,2)$, which are not comparable.

Let $L_N(j(p))$ be the set of partitions of N such that the cardinalities of its blocks consist of the partition of integer n into $j(p)$. For $j(p) = (j_1 \cdots j_p)$, let e_i be the number of elements equal to i; then $j(p)$ can be expressed by

$e = 1^{e_1}2^{e_2}\dots n^{e_n}$, which is called the type of $\pi \in L_N(j(p))$, type(π), in [1, p. 70].

It is well known that the cardinality of $L_N(j(p))$ is given by

$$b(j(p); n) = \frac{n!}{j_1! \cdots j_p! e_1! \cdots e_p!} \tag{2}$$

(cf. [2, Theorem 13.2]).

For a partition $\pi = (N_1, \dots, N_p)$ of N, let

$$S(\pi) = \prod_{t=1}^{p} \sum_{j=1}^{m} \prod_{i \in N_t} a_{ij},$$

and also let

$$T(\pi) = \sum \prod_{t=1}^{p} \prod_{i \in N_t} a_{il_t},$$

where the summation is for all p-permutations $(l_1 \dots l_p)$ of M. Note that in $S(\pi)$ and $T(\pi)$ there are m^p and $m(m-1)\cdots(m-p+1)$ terms respectively.

For $j(p) \in G(p)$, let

$$S(j(p)) = \sum_{\pi \in L_N(j(p))} S(\pi), \tag{3}$$

and also let

$$T(j(p)) = \sum_{\pi \in L_N(j(p))} T(\pi). \tag{4}$$

Then $T(j(p))$ is the sum of the coefficients of monomials of the form $x_{l_1}^{j_1} \cdots x_{l_p}^{j_p}$ in the polynomial (1). Note that the permanent of the matrix A, Per(A), is $T(0)$ or equivalently $T(j(n))$ in our notation.

3. A GENERALIZATION OF THE BINET-MINC FORMULA

For a partition σ with q blocks, let π be a refinement of σ, and also let the ith block of σ be the union of h_i blocks of π. It is well known that the

Möbius function $\mu(\pi,\sigma)$ is given by

$$\mu(\pi,\sigma)=\prod_{i=1}^{q}(-1)^{h_i-1}(h_i-1)!. \tag{5}$$

LEMMA. *For $\pi\in L_N$, we have*

$$T(\pi)=\sum_{\pi\leqslant\sigma\in L_N}\mu(\pi,\sigma)S(\sigma). \tag{6}$$

Proof. It is clear, from the definition of $S(\pi)$ and $T(\pi)$, that we have

$$S(\pi)=\sum_{\sigma\geqslant\pi}T(\sigma). \tag{7}$$

Applying the Möbius inversion formula (cf. [1, p. 152]), we have (6). ■

For $\pi\geqslant\sigma$, type$[\pi,\sigma]$ is defined as $g=1^{g_1}2^{g_2}\dots p^{g_p}$, where g_i is the number of blocks of σ made up of exactly i blocks of π (cf. [1, p. 70]). Thus the Möbius function (5) becomes

$$\mu(\pi,\sigma)=\prod_{i=1}^{n}\left[(-1)^{i-1}(i-1)!\right]^{g_i}.$$

THEOREM 1. *Let $k(q)\geqslant j(p)$. For a given type g and any $\sigma\in L_N(k(q))$, the cardinality of the set $\{\pi:\pi\in L_N(j(p))$, type$[\pi,\sigma]=g\}$ is free from σ and depends only on $j(p)$, $k(q)$, and g. The cardinality of the set $\{\pi:\pi\in L_N(j(p)),\ \pi\leqslant\sigma\}$ is also free from σ and depends only on $j(p)$ and $k(q)$.*

Proof. For any σ_1,σ_2 in $L_N(k(q))$, there exists a permutation of the set N which gives a bijection between the set $\{\pi:\pi\in L_N(j(p))$, type$[\pi,\sigma_1]=g\}$ and the set $\{\pi:\pi\in L_N(j(p))$, type$[\pi,\sigma_2]=g\}$. The remaining part follows immediately, since the set $\{\pi:\pi\in L_N(j(p)),\ \pi\leqslant\sigma\}$ is the union of the sets $\{\pi:\pi\in L_N(j(p))$, type$[\pi,\sigma]=g\}$ for all possible types g. ■

This theorem enables us to introduce the symbols $b(j(p);k(q),g)$ and $b(j(p);k(q))$, which represent the cardinalities of the sets in the theorem. The derivation of an expression for $b(j(p);k(q),g)$ is fairly complicated, and is given in the appendix together with examples of the expression. Similarly, let $b^*(k(q);j(p),g)$ and $b^*(k(q);j(p))$ be the cardinalities of the sets $\{\sigma:\sigma\in L_N(k(q))$, type$[\pi,\sigma]=g\}$ and $\{\sigma:\sigma\in L_N(k(q)),\ \sigma\geqslant\pi\}$ for $\pi\in$

$L_N(j(p))$, respectively. Counting the number of pairs in the set $\{(\pi, \sigma): \pi \leq \sigma, \pi \in L_N(j(p)), \sigma \in L_N(k(q)), \text{type}[\pi, \sigma] = g\}$ and also in the set $\{(\pi, \sigma): \pi \leq \sigma, \pi \in L_N(j(p)), \sigma \in L_N(k(q))\}$, we have the following corollary.

COROLLARY. *For $k(q) \geq j(p)$, we have*

$$\frac{b(j(p); k(q), g)}{b^*(k(q); j(p), g)} = \frac{b(j(p); k(q))}{b^*(k(q); j(p))} = \frac{b(j(p); n)}{b(k(q); n)}. \tag{8}$$

Note that for $q = 1$, the only member in $L_N(k(1))$ is $\sigma = I$, and $\text{type}[\pi, I] = 1^0 2^0 \dots p^1 =: (p)$ for all $\pi \in L_N(j(p))$. Since $b^*(k(1); j(p), (p)) = b^*(k(1); j(p)) = b(k(1); n) = 1$ and $b(j(p); k(1), (p)) = b(j(p); k(1)) = b(j(p); n)$, (8) may be regarded as a generalization of (2). Another generalization of (2) is given in Good [4, Lemma, p. 334].

We give an identity corresponding to (7) for $S(j(p))$.

THEOREM 2. *For $j(p) \in G(p)$, we have*

$$S(j(p)) = \sum_{q=1}^{p} \sum_{k(q) \geq j(p)} b(j(p); k(q)) T(k(q)). \tag{9}$$

Proof. From the definition and (7), we have

$$S(j(p)) = \sum_{\pi \in L_N(j(p))} \sum_{\sigma \geq \pi} T(\sigma). \tag{10}$$

By changing the order of the summation and by using the zeta function $\zeta(\pi, \sigma)$, the right hand side of (10) becomes

$$\sum_{q=1}^{p} \sum_{k(q) \geq j(p)} \sum_{\sigma \in L_N(k(q))} \sum_{\pi \in L_N(j(p))} \zeta(\pi, \sigma) T(\sigma).$$

From Theorem 1, (9) follows. ∎

EXAMPLE. Let $n = 6$, $p = 3$, and $j(3) = (321)$. The expressions for $S(j(3))$ and $S(k(2))$ with $k(2) \geq j(3)$ and $S(k(1))$ in Theorem 2 are given as follows:

$$\begin{aligned}
S(321) &= T(321) + 10T(51) + 4T(42) + 6T(33) + 60T(6),\\
S(51\ \ ) &= T(51) + 6T(6),\\
S(42\ \ ) &= T(42) + 15T(6),\\
S(33\ \ ) &= T(33) + 10T(6),\\
S(6\ \ \ \ ) &= T(6).
\end{aligned}$$

Solving this equation, we have

$$T(321) = S(321) - \{10S(51) + 4S(42) + 6S(33)\} + 120S(6).$$

As is seen in the example, the equations (9) for $j(p) \in G(p)$ and all $k(q) \in G(q)$, $q = 1, \ldots, p-1$, are simultaneous linear equations in $T(j(p))$ and $T(k(q))$ for $q = 1, \ldots, p-1$. Their solution is given in the following theorem.

THEOREM 3. *Putting*

$$c(j(p); k(q)) = (-1)^{p-q} \sum_{g} \prod_{i=1}^{p} [(i-1)!]^{g_i} b(j(p); k(q), g),$$

we have for $j(p) \in G(p)$

$$T(j(p)) = \sum_{q=1}^{p} \sum_{k(q) \geq j(p)} c(j(p); k(q)) S(k(q)), \tag{11}$$

where the summation $\sum_g$ *is for all possible types* $g = \text{type}[\pi, \sigma]$.

Proof. From (4) and (6), we have

$$T(j(p)) = \sum_{\pi \in L_N(j(p))} \sum_{\pi \leq \sigma \in L_N} \mu(\pi, \sigma) S(\sigma). \tag{12}$$

Changing the order of the summation, the right hand side of (12) becomes

$$\sum_{q=1}^{p} \sum_{k(q)\in G(q)} \sum_{\sigma \in L_N(k(q))} S(\sigma) \sum_{\pi \in L_N(j(p)),\, \pi \leqslant \sigma} \mu(\pi,\sigma).$$

Since the Möbius function $\mu(\pi,\sigma)$ is determined by type$[\pi,\sigma]$, and $\sum_{i=1}^{p}(i-1)g_i = p-q$, Theorem 1 leads us to (11). ■

Let $k(q)=(k_1\dots k_q)\in G(q)$. When $p=n$, we have $T(j(n))=\operatorname{Per}(A)$ and $c(j(n);k(q))=(-1)^{n-q}\prod_{i=1}^{q}(k_i-1)!$. Hence from the theorem we have the following corollary, which is the well-known Binet-Minc formula [7].

COROLLARY.

$$\operatorname{Per}(A) = \sum_{q=1}^{n} \sum_{k(q)\in G(q)} c(k(q))S(k(q)),$$

where $k(q)=(k_1\dots k_q)\in G(q)$ *and*

$$c(k(q)) = (-1)^{n-q}\prod_{i=1}^{q}(k_i-1)!.$$

4. EXPRESSION FOR THE STIRLING NUMBERS

As an application of Theorems 2 and 3, we give two expressions for the Stirling numbers. Let $a_{ij}=1$, $i=1,\dots,n$, $j=1,\dots,m$, and $n\leqslant m$. According to the definitions of $S(j(p))$, $T(j(p))$, and $b(j(p);n)$, we have

$$S(j(p)) = b(j(p);n)m^p,$$

$$T(j(p)) = b(j(p);n)(m)_p,$$

where $(m)_p = m(m-1)\cdots(m-p+1)$. From the definition of the Stirling numbers of the second kind, $S(p,q)$, it follows from Theorem 2 that

$$S(p,q) = \frac{1}{b(j(p);n)} \sum_{k(q)\geqslant j(p)} b(j(p);k(q))b(k(q);n).$$

Similarly, for the Stirling numbers of the first kind, $s(p,q)$, we have from Theorem 3

$$s(p,q)=\frac{1}{b(j(p);n)}\sum_{k(q)\geqslant j(p)}c(j(p);k(q))b(k(q);n).$$

In the particular case of $p=n$ [i.e. $j(n)=(1\dots1)$], these formulas are found in Riordan [9, p. 195] and Andrews [2, Theorem 13.8].

APPENDIX. AN EXPRESSION FOR $b(j(p);k(q),g)$

Let $j(p)=(j_1\cdots j_p)$, and its corresponding type be $e=1^{e_1}2^{e_2}\dots n^{e_n}$. Furthermore, let $g=1^{g_1}2^{g_2}\dots p^{g_p}$ with $g_1+g_2+\cdots+g_p=q$ and $\sum_{i=1}^{p}ig_i=p$.

The problem is to obtain the cardinality of the set of π's with type$(\pi)=e$, i.e. $\pi\in L_N(j(p))$ and type$[\pi,\sigma]=g$, given that $\sigma\in L_N(k(q))$. The set of these π's is not empty for some g if $j(p)\leqslant k(q)$.

Let the blocks of σ be $B_1, B_2,\dots, B_q$. Then $\pi\leqslant\sigma$ in type$[\pi,\sigma]=g$ means that B_1 is subdivided into l_1 subsets, say $A_{11},\dots,A_{1l_1}$, B_2 into $A_{21},\dots,A_{2l_2}$, and so on, and finally B_q into $A_{q1},\dots,A_{ql_q}$, so that blocks of π are $A_{11},\dots,A_{1l_1},A_{21},\dots,A_{2l_2},\dots,A_{q1},\dots,A_{ql_q}$ for some $(l_1,\dots,l_q)$, and type$[\pi,\sigma]=g=1^{g_1}2^{g_2}\dots p^{g_p}$, where g_i is the number of $l_1,\dots,l_q$ equal to i $(i=1,\dots,p)$. Note that $(l_1,\dots,l_q)$ becomes, if rearranged in nondecreasing order, a partition of the integer p into q parts. Corresponding to the pair (π,σ), there exists a set of permutations of the form

$$\lambda=\left(\lambda_{11}\dots\lambda_{1l_1}\lambda_{21}\dots\lambda_{2l_2}\dots\lambda_{q1}\dots\lambda_{ql_q}\right)$$

of $\{1,\dots,p\}$ with

$$k_i=j_{\lambda_{i1}}+\cdots+j_{\lambda_{il_i}}\qquad(i=1,\dots,q).$$

Now in order to avoid irrelevant repetitions, we impose the following conditions:

(a) $j_{\lambda_{i1}}\geqslant\cdots\geqslant j_{\lambda_{il_i}}$;
(b) if $s<t$ and $j_{\lambda_{is}}=j_{\lambda_{it}}$, then $\lambda_{is}<\lambda_{it}$;
(c) if $i_1<i_2$ and $j_{\lambda_{i_1s}}=j_{\lambda_{i_2t}}$, then $\lambda_{i_1s}<\lambda_{i_2t}$.

We call a permutation $\lambda = (\lambda_{11} \dots \lambda_{1l_1}\lambda_{21} \dots \lambda_{2l_2} \dots \lambda_{q1} \dots \lambda_{ql_q})$ satisfying (a), (b), and (c) *representative*. Under these conditions, $j^{(i)}(l_i) := (j_{\lambda_{i1}} \cdots j_{\lambda_{il_i}})$ is again a partition of integers k_i into l_i parts $(i = 1, \dots, q)$. Thus, for each representative permutation, the number of π's with type$(\pi) = e$ and type$[\pi, \sigma] = g$ is, from (9), $\prod_{i=1}^{q} b(j^{(i)}(l_i); k_i)$. Hence we have

$$b(j(p); k(q); g) = \sum \prod_{i=1}^{q} b(j^{(i)}(l_i); k_i), \tag{13}$$

where the summation is for all representative permutations.

EXAMPLE. Let $n = 6$, $p = 3$, and $q = 2$. Furthermore, let $k(2) = (k_1 k_2) = (42)$ and $(j_1 j_2 j_3) = (321)$. Then $(k_1 k_2) = (42)$ is obtained from $(j_1 j_2 j_3) = (321)$ as follows: $(42) = (3+12)$, i.e., $k_1 = j_1 + j_3$ and $k_2 = j_2$. We have $(l_1 l_2) = (21)$, and the corresponding representative permutation of $\{1,2,3\}$ is $(13{:}2)$, where colon is used to emphasize the fact $l_1 = 2$ and $l_2 = 1$. From (13) we obtain

$$b(j(3); k(2), g = 1^1 2^1 3^0) = b((31);4) \cdot b((2);2) = 4.$$

For a small value of n, there exists none or only one representative permutation corresponding to a triplet $j(p), k(q), g$. When $n = 14$, $p = 7$, $j(7) = (3322211)$, $q = 3$, and $k(3) = (644)$, there are two representative permutations. In this case

$$6 = 3+2+1, \quad 4 = 3+1, \quad 4 = 2+2 \quad \text{and}$$

$$6 = 2+2+2, \quad 4 = 3+1, \quad 4 = 3+1$$

and both are of the type $g = 1^0 2^2 3^1$. Hence the two representative permutations are

$$(136{:}27{:}45) \quad \text{and} \quad (345{:}16{:}27).$$

Furthermore, from (13) we obtain

$$\begin{aligned} b(j(7); k(3), g = 1^0 2^2 3^1) &= b((321);6) \cdot b((31);4) \cdot b((22);4) \\ &\quad + b((222);6) \cdot b((31);4)^2 \\ &= 960. \end{aligned}$$

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